(2+1) null-plane quantum Poincaré group from a factorized universal R-matrix

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Abstract

The non-standard (Jordanian) quantum deformations of $so(2,2)$ and (2+1) Poincaré algebras are constructed by starting from a quantum $sl(2,\mathbb{R})$ basis such that simple factorized expressions for their corresponding universal R-matrices are obtained. As an application, the null-plane quantum (2+1) Poincaré Poisson-Lie group is quantized by following the FRT prescription. Matrix and differential representations of this null-plane deformation are presented, and the influence of the choice of the basis in the resultant q-Schrödinger equation governing the deformed null plane evolution is commented.

1 Introduction

Among quantum deformations of Poincaré algebra we find three remarkable Hopf structures. Two of them are obtained in a natural way within a purely kinematical framework encoded within the usual Poincaré basis. They are the well-known κ-Poincaré algebra [1, 2] where the deformation parameter can be interpreted as a fundamental time scale and a q-Poincaré algebra [3] where the quantum parameter is a fundamental length. On the other hand, the remaining structure (the null-plane quantum Poincaré algebra recently introduced in [4, 5]) strongly differs from the previous ones: firstly, it is constructed in a null-plane context where the Poincaré invariance splits into a kinematical and dynamical part [6] and, secondly, this case is a quantization of a non-standard (triangular) coboundary Lie bialgebra.

We also recall that the related problem of obtaining universal R-matrices for standard Poincaré deformations has been only solved in (2+1) dimensions [7, 8]. In the non-standard case, relevant successes have been recently obtained: the universal R-matrix for the (1+1) case has been deduced in [9, 10, 11] and a (2+1) solution has been recently given in [12].

The aim of this letter is twofold: on one hand, to present a simplified construction of the universal R-matrices for non-standard quantum $so(2,2)$ and (2+1) Poincaré algebras, which is based on the $sl(2,\mathbb{R})$ factorized R-matrix introduced in [13] (sections 2 and 3). Whithin this construction a (non-linear) change of basis (whose origin lies in a T-matrix construction [11, 14]) is essential, and will lead to rather new expressions for all these quantizations. From a physical point of view, it is important to stress that $so(2,2)$ is interpreted in a conformal context: i.e., its classical counterpart acts as the group of conformal transformations on the (1+1) Minkowskian space-time. The second objective is to get a deeper insight in the (2+1) null-plane quantization by constructing the associated quantum group and by exploring its representation theory. All these results are presented in Section 4.

2 Non-standard quantum $so(2,2)$ revisited

Let us consider the coproduct and the commutation relations of the non-standard quantum $sl(2,\mathbb{R})$, denoted by $U_z sl(2,\mathbb{R}) = \langle A, A_+, A_- \rangle$, as [13]:

$$\begin{aligned} \Delta(A_+) &= 1 \otimes A_+ + A_+ \otimes 1, \\ \Delta(A) &= 1 \otimes A + A \otimes e^{2zA_+}, \\ \Delta(A_-) &= 1 \otimes A_- + A_- \otimes e^{2zA_+}, \end{aligned} \tag{2.1}$$

$$[A, A_+] = \frac{e^{2zA_+} - 1}{z}, \quad [A, A_-] = -2A_- + zA^2, \quad [A_+, A_-] = A, \tag{2.2}$$

and the quantum Casimir belonging to the centre of $U_z sl(2,\mathbb{R})$ given by:

$$\mathcal{C}_z = \frac{1}{2} A\, e^{-2zA_+} A + \frac{1 - e^{-2zA_+}}{2z} A_- + A_- \frac{1 - e^{-2zA_+}}{2z} + e^{-2zA_+} - 1, \tag{2.3}$$

These relations are obtained from the ones given in [4, 15] in terms of $\{J_3, J_+, J_-\}$, by means of the change of basis [13]

$$
\begin{aligned}
&A_+ = J_+, \qquad A = e^{zJ_+} J_3, \\
&A_- = e^{zJ_+} J_- - \frac{z}{4} e^{zJ_+} \sinh(zJ_+).
\end{aligned}
\tag{2.4}
$$

In [13] it is shown that the universal element

$$
R_z = \exp\{-zA_+ \otimes A\} \exp\{zA \otimes A_+\} \tag{2.5}
$$

is a solution of the quantum Yang–Baxter equation and also verifies the property

$$
\sigma \circ \Delta(X) = \mathcal{R}_z \Delta(X) \mathcal{R}_z^{-1}, \qquad \forall\, X \in U_z sl(2, \mathbb{R}), \tag{2.6}
$$

being σ the flip operator $\sigma(a \otimes b) = b \otimes a$. Hence, R_z is the quantum universal R-matrix for $U_z sl(2, \mathbb{R})$.

Two comments concerning this universal R-matrix are in order: firstly, the significant simplification obtained (as far as the commutation rules (2.2) are concerned) with respect to the original formulation of this non-standard $sl(2, \mathbb{R})$ deformation. Secondly, we recall that the factorized expression (2.5) comes from a universal T-matrix formalism [11, 14]. From this point of view, the interest of finding such a kind of factorized expressions is directly related to the interpretation of the transfer matrices as quantum monodromies and the obtention of more manageable algebraic models in quantum field theory (see [16]).

Let us now consider two copies of the non-standard quantum $sl(2, \mathbb{R})$ algebra, the former with z and the latter with $-z$ as deformation parameters: $U_z^{(1)} sl(2, \mathbb{R}) = \langle A^1, A^1_+, A^1_- \rangle$ and $U_{-z}^{(2)} sl(2, \mathbb{R}) = \langle A^2, A^2_+, A^2_- \rangle$. The generators defined by

$$
\begin{aligned}
&K = \frac{1}{2}(A^1 - A^2), \qquad D = \frac{1}{2}(A^1 + A^2), \\
&H = A^1_+ + A^2_+, \qquad P = A^1_+ - A^2_+, \\
&C_1 = -A^1_- - A^2_-, \qquad C_2 = A^1_- - A^2_-,
\end{aligned}
\tag{2.7}
$$

give rise to a non-standard quantum deformation of $so(2,2)$ [4]:

$$
U_z so(2,2) \simeq U_z^{(1)} sl(2, \mathbb{R}) \oplus U_{-z}^{(2)} sl(2, \mathbb{R}).
$$

At a purely classical level, $SO(2,2)$ can be regarded in this basis as the group of conformal transformations of the (1+1) Minkowskian space-time, where K generates the boosts, H the time translations, P the space translations, D is a dilation generator and C_1, C_2 generate specific conformal transformations. The Hopf algebra structure of $U_z so(2,2)$ obtained in this way is given by the following coproduct (Δ), counit (ϵ), antipode (γ) and commutation relations:

$$
\begin{aligned}
&\Delta(H) = 1 \otimes H + H \otimes 1, \qquad \Delta(P) = 1 \otimes P + P \otimes 1, \\
&\Delta(K) = 1 \otimes K + K \otimes e^{zP} \cosh zH + D \otimes e^{zP} \sinh zH,
\end{aligned}
$$

$$
\begin{aligned}
\Delta(D) &= 1 \otimes D + D \otimes e^{zP} \cosh zH + K \otimes e^{zP} \sinh zH, \\
\Delta(C_1) &= 1 \otimes C_1 + C_1 \otimes e^{zP} \cosh zH - C_2 \otimes e^{zP} \sinh zH, \\
\Delta(C_2) &= 1 \otimes C_2 + C_2 \otimes e^{zP} \cosh zH - C_1 \otimes e^{zP} \sinh zH,
\end{aligned}
\tag{2.8}
$$

$$
\epsilon(X) = 0, \qquad \text{for } X \in \{K, H, P, C_1, C_2, D\}, \tag{2.9}
$$

$$
\begin{aligned}
&\gamma(H) = -H, \qquad \gamma(P) = -P, \\
&\gamma(K) = -Ke^{-zP} \cosh zH + De^{-zP} \sinh zH, \\
&\gamma(D) = -De^{-zP} \cosh zH + Ke^{-zP} \sinh zH, \\
&\gamma(C_1) = -C_1 e^{-zP} \cosh zH - C_2 e^{-zP} \sinh zH, \\
&\gamma(C_2) = -C_2 e^{-zP} \cosh zH - C_1 e^{-zP} \sinh zH,
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&[K, H] = \frac{1}{z}(e^{zP} \cosh zH - 1), \qquad [K, P] = \frac{1}{z} e^{zP} \sinh zH, \\
&[K, C_1] = C_2 - z(K^2 + D^2), \qquad [K, C_2] = C_1 + 2zKD, \\
&[D, H] = \frac{1}{z} e^{zP} \sinh zH, \qquad [D, P] = \frac{1}{z}(e^{zP} \cosh zH - 1), \\
&[D, C_1] = -C_1 - 2zKD, \qquad [D, C_2] = -C_2 + z(K^2 + D^2), \\
&[H, C_1] = -2D, \quad [H, C_2] = 2K, \quad [P, C_1] = -2K, \quad [P, C_2] = 2D, \\
&[K, D] = 0, \qquad [H, P] = 0, \qquad [C_1, C_2] = 0.
\end{aligned}
\tag{2.11}
$$

Note that (2.8) presents an interesting feature: both generators H and P are primitive ones. Therefore, this conformal approach to $so(2,2)$ leads to a quantum structure that can be interpreted as an attemp in order to deform the (1+1) Minkowskian space and time in a rather symmetrical way.

Two elements of the centre of $U_z so(2,2)$ are constructed from the quantum Casimirs (2.3) of $U_z^{(1)} sl(2, \mathbb{R})$ and $U_{-z}^{(2)} sl(2, \mathbb{R})$ as:

$$
\mathcal{C}_1^q = \mathcal{C}_z^{(1)} + \mathcal{C}_{-z}^{(2)}, \qquad \mathcal{C}_2^q = \mathcal{C}_z^{(1)} - \mathcal{C}_{-z}^{(2)}. \tag{2.12}
$$

After a straightforward computation we get:

$$
\begin{aligned}
\mathcal{C}_1^q = {} & Ke^{-zP} \cosh(zH) K + De^{-zP} \cosh(zH) D \\
& - Ke^{-zP} \sinh(zH) D - De^{-zP} \sinh(zH) K \\
& + \frac{1}{2z}(1 - e^{-zP} \cosh zH) C_2 + C_2 \frac{1}{2z}(1 - e^{-zP} \cosh zH) \\
& - e^{-zP} \frac{\sinh zH}{2z} C_1 - C_1 e^{-zP} \frac{\sinh zH}{2z} + 2(e^{-zP} \cosh zH - 1),
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
\mathcal{C}_2^q = {} & Ke^{-zP} \cosh(zH) D + De^{-zP} \cosh(zH) K \\
& - Ke^{-zP} \sinh(zH) K - De^{-zP} \sinh(zH) D \\
& - \frac{1}{2z}(1 - e^{-zP} \cosh zH) C_1 - C_1 \frac{1}{2z}(1 - e^{-zP} \cosh zH) \\
& + e^{-zP} \frac{\sinh zH}{2z} C_2 + C_2 e^{-zP} \frac{\sinh zH}{2z} - 2e^{-zP} \sinh zH.
\end{aligned}
\tag{2.14}
$$

Likewise, the universal R-matrix for $U_z so(2,2)$ can be easily deduced as a product of those corresponding to the two copies of $U_z sl(2,\mathbb{R})$ (2.5):

$$
\begin{aligned}
\mathcal{R}_z = R_z^{(1)} R_{-z}^{(2)} &= \exp\{-zA_+^1 \otimes A^1\}\exp\{zA^1 \otimes A_+^1\}\exp\{zA_+^2 \otimes A^2\}\exp\{-zA^2 \otimes A_+^2\} \\
&= \exp\{-zA_+^1 \otimes A^1 + zA_+^2 \otimes A^2\}\exp\{zA^1 \otimes A_+^1 - zA^2 \otimes A_+^2\} \\
&= \exp\{-z(H \otimes K + P \otimes D)\}\exp\{z(K \otimes H + D \otimes P)\},
\end{aligned}
$$

which can be finally written in a complete "factorized" form as:

$$
\mathcal{R}_z = \exp\{-zH \otimes K\}\exp\{-zP \otimes D\}\exp\{zD \otimes P\}\exp\{zK \otimes H\}. \tag{2.15}
$$

The first order in z gives the classical r-matrix

$$
r = z(K \wedge H + D \wedge P), \tag{2.16}
$$

which, as expected, is a solution of the classical Yang–Baxter equation.

3 (2+1) null-plane quantum Poincaré algebra

3.1 Null-plane classical Poincaré algebra

We briefly describe the classical structure of the (2+1) Poincaré algebra $\mathcal{P}(2+1)$ in relation with the null-plane evolution scheme [17], in which the initial state of a quantum relativistic system can be defined on a light-like plane Π_n^τ defined by $n \cdot x = \tau$, where n is a light-like vector and τ a real constant. In particular, if $n = (\frac{1}{2}, 0, \frac{1}{2})$ and the coordinates

$$
x^- = n \cdot x = \frac{1}{2}(x^0 - x^2), \qquad x^+ = x^0 + x^2, \tag{3.1}
$$

are considered, a point $x \in \Pi_n^\tau$ will be labelled by (x^+, x^1) while the remaining one (x^-) plays the role of a time parameter τ. A basis $\{P_+, P_1, P_-, E_1, F_1, K_2\}$ of the (2+1) Poincaré algebra consistent with these coordinates is provided by the generators P_+, P_-, E_1 and F_1 which are defined in the terms of the usual kinematical ones $\{P_0, P_1, P_2, K_1, K_2, J\}$ by:

$$
P_+ = \frac{1}{2}(P_0 + P_2), \quad P_- = P_0 - P_2, \quad E_1 = \frac{1}{2}(K_1 + J), \quad F_1 = K_1 - J. \tag{3.2}
$$

This "null-plane" basis has the following non-vanishing commutation rules:

$$
\begin{aligned}
&[K_2, P_\pm] = \pm P_\pm, \quad [K_2, E_1] = E_1, \quad [K_2, F_1] = -F_1, \\
&[E_1, P_1] = P_+, \quad [F_1, P_1] = P_-, \quad [E_1, F_1] = K_2, \\
&[P_+, F_1] = -P_1, \quad [P_-, E_1] = -P_1.
\end{aligned} \tag{3.3}
$$

The operators $\{P_+, P_1, E_1, K_2\}$ are the infinitesimal generators of the stability group S_+ of the null-plane Π_n^0 $(\tau = 0)$. The remaining generators have a dynamical

significance: the hamiltonian P_- translates Π_n^0 into Π_n^τ and F_1 rotates it around the surface of the light-cone $x^1 = 0$.

Finally, let us recall that the centre of $\mathcal{P}(2+1)$ is generated by the square of the mass operator M^2 and the intrinsic angular momentum L, which read:

$$M^2 = 2P_-P_+ - P_1^2, \tag{3.4}$$
$$L = K_2P_1 + E_1P_- - F_1P_+. \tag{3.5}$$

3.2 Null-plane quantum Poincaré algebra

This null-plane Poincaré algebra is naturally linked by a contraction procedure to $so(2,2)$ when the latter is written in a basis of the kind (2.7). The explicit form of that contraction mapping is as follows:

$$\begin{aligned}
&P_+ = \varepsilon\,\frac{1}{\sqrt{2}}P, \quad P_1 = \varepsilon\,K, \quad P_- = -\varepsilon\,\frac{1}{\sqrt{2}}C_2,\\
&E_1 = -\frac{1}{\sqrt{2}}H, \quad F_1 = \frac{1}{\sqrt{2}}C_1, \quad K_2 = D.
\end{aligned} \tag{3.6}$$

In the quantum case, the deformation parameter has also to be transformed as $w = \frac{1}{\varepsilon\sqrt{2}}z$. Therefore, by applying (3.6) in the Hopf algebra of $U_z so(2,2)$ (2.8–2.11) and then by making the limit $\varepsilon \to 0$ we get the resulting Hopf structure for the quantum (2+1) null plane Poincaré algebra $U_w\mathcal{P}(2+1)$:

$$\begin{aligned}
&\Delta(P_+) = 1\otimes P_+ + P_+\otimes 1, \qquad \Delta(E_1) = 1\otimes E_1 + E_1\otimes 1,\\
&\Delta(P_-) = 1\otimes P_- + P_-\otimes e^{2wP_+}, \qquad \Delta(P_1) = 1\otimes P_1 + P_1\otimes e^{2wP_+},\\
&\Delta(F_1) = 1\otimes F_1 + F_1\otimes e^{2wP_+} - 2wP_-\otimes e^{2wP_+}E_1,\\
&\Delta(K_2) = 1\otimes K_2 + K_2\otimes e^{2wP_+} - 2wP_1\otimes e^{2wP_+}E_1,
\end{aligned} \tag{3.7}$$

$$\epsilon(X) = 0, \qquad \text{for } X \in \{K_2, P_+, P_-, P_1, E_1, F_1\}, \tag{3.8}$$

$$\begin{aligned}
&\gamma(P_+) = -P_+, \qquad \gamma(E_1) = -E_1,\\
&\gamma(P_-) = -P_-e^{-2wP_+}, \quad \gamma(P_1) = -P_1e^{-2wP_+},\\
&\gamma(F_1) = -F_1e^{-2wP_+} - 2wP_-e^{-2wP_+}E_1,\\
&\gamma(K_2) = -K_2e^{-2wP_+} - 2wP_1e^{-2wP_+}E_1,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&[K_2, P_+] = \frac{1}{2w}(e^{2wP_+} - 1), \qquad [K_2, P_-] = -P_- - wP_1^2,\\
&[K_2, E_1] = E_1e^{2wP_+}, \qquad [K_2, F_1] = -F_1 - 2wP_1K_2,\\
&[E_1, P_1] = \frac{1}{2w}(e^{2wP_+} - 1), \qquad [F_1, P_1] = P_- + wP_1^2,\\
&[E_1, F_1] = K_2, \qquad [P_+, F_1] = -P_1, \qquad [P_-, E_1] = -P_1,
\end{aligned} \tag{3.10}$$

where the remaining commutators are zero. Note that the generators of the null-plane stability group close a Hopf subalgebra U_wS_+. As a byproduct of the original

change of basis within $sl(2,\mathbb{R})$, these commutation rules are simpler than the ones given in [4].

The quantum Casimirs belonging to the centre of $U_w\mathcal{P}(2+1)$ are deduced from (2.13) and (2.14) by means of the limits:

$$M_q^2 = \lim_{\varepsilon\to 0}(-\varepsilon^2 \mathcal{C}_1^q), \qquad L_q = \frac{1}{2}\lim_{\varepsilon\to 0}(\varepsilon \mathcal{C}_2^q), \tag{3.11}$$

explicitly,

$$M_q^2 = P_- \frac{1-e^{-2wP_+}}{w} - P_1^2 e^{-2wP_+}, \tag{3.12}$$

$$L_q = K_2 P_1 e^{-2wP_+} + E_1(P_- + wP_1^2)e^{-2wP_+} - F_1 \frac{1-e^{-2wP_+}}{2w}. \tag{3.13}$$

The universal R-matrix $U_w\mathcal{P}(2+1)$ is also directly obtained from (2.15) and reads:

$$\mathcal{R}_w = \exp\{2wE_1\otimes P_1\}\exp\{-2wP_+\otimes K_2\}\exp\{2wK_2\otimes P_+\}\exp\{-2wP_1\otimes E_1\}. \tag{3.14}$$

A differential representation of U_wS_+ with coordinates (p_+, p_1) can be given as follows:

$$P_+ = p_+, \quad P_1 = p_1, \quad K_2 = \frac{e^{2wp_+}-1}{2w}\partial_+, \quad E_1 = \frac{e^{2wp_+}-1}{2w}\partial_1, \tag{3.15}$$

where $\partial_+ = \frac{\partial}{\partial p_+}$ and $\partial_1 = \frac{\partial}{\partial p_1}$. With the aid of the quantum Casimirs a spin-zero differential representation ($L_q = 0$) for the two remaining generators of $U_w\mathcal{P}(2+1)$ can be deduced:

$$P_- = \frac{w(m_q^2 + p_1^2 e^{-2wp_+})}{1-e^{-2wp_+}}, \quad F_1 = p_1\partial_+ + \frac{w(m_q^2+p_1^2)}{1-e^{-2wp_+}}\partial_1, \tag{3.16}$$

where m_q^2 is the eigenvalue of the q-Casimir (3.12). Similarly to the classical case, we can take the coordinate x^- as an evolution parameter (τ) and thus we can consider a wave function $\psi(p_+, p_1, \tau)$ whose evolution is determined by the q-Scrödinger equation provided by the Hamiltonian P_-: $i\partial_\tau\psi = P_-\psi$. In terms of the representation (3.16) we get:

$$i\partial_\tau\psi(p_+, p_1, \tau) = \frac{w(m_q^2 + p_1^2 e^{-2wp_+})}{1-e^{-2wp_+}}\psi(p_+, p_1, \tau), \tag{3.17}$$

which is different from the one given in [5] for the (3+1) case. This fact can be more clearly appreciated by writing the power series expansion in w of P_-:

$$\begin{aligned} P_- &= \frac{w(m_q^2 + p_1^2 e^{-2wp_+})}{1-e^{-2wp_+}} = \frac{w(m_q^2 e^{wp_+} + p_1^2 e^{-wp_+})}{2\sinh wp_+} \\ &= \frac{m_q^2+p_1^2}{2p_+} + w\frac{m_q^2-p_1^2}{2} + w^2\frac{p_+(m_q^2+p_1^2)}{6} + o(w^3). \end{aligned} \tag{3.18}$$

The zero-term in w can be identified with a kinetic term of the null-plane bound state equation in quantum chromodynamics [18, 19] while all remaining terms in w constitute a dynamical part, now including a first order term in w (that is absent in [5]). Therefore, this deformation of the null-plane symmetry has some intrinsic dynamical content whose explicit description depends on the way in which the deformation is constructed.

4 (2+1) Null-plane quantum Poincaré group

The Lie bialgebra underlying the quantum Hopf algebra of $U_w\mathcal{P}(2+1)$ is generated by the non-standard classical r-matrix (first order in w of (3.14)):

$$r = 2(K_2 \wedge P_+ + E_1 \wedge P_1), \tag{4.1}$$

which provides the cocommutators $\delta(X) = [1 \otimes X + X \otimes 1, r]$:

$$\begin{array}{l} \delta(P_+) = 0, \qquad \delta(E_1) = 0, \\ \delta(P_1) = 2P_1 \wedge P_+, \qquad \delta(P_-) = 2P_- \wedge P_+, \\ \delta(F_1) = 2(F_1 \wedge P_+ + E_1 \wedge P_-), \\ \delta(K_2) = 2(K_2 \wedge P_+ + E_1 \wedge P_1). \end{array} \tag{4.2}$$

They are related to the first order term in the deformation parameter of the coproduct (3.7) by means of $\delta = \Delta_{(1)} - \sigma \circ \Delta_{(1)}$.

The r-matrix (4.1) also allows to deduce the associated Poisson structure to the Poincaré algebra. Let the four-dimensional matrix representation of $\mathcal{P}(2+1)$ given by:

$$D(P_+) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \quad D(P_-) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad D(P_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$D(E_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \quad D(F_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad D(K_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.3}$$

Then a 4×4 representation of the element $g = e^{a^+ P_+} e^{a^- P_-} e^{a^1 P_1} e^{e^1 E_1} e^{f^1 F_1} e^{k^2 K_2}$ belonging to the (2+1) Poincaré group is

$$D(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{a^+}{2} + a^- & \Lambda_0^0 & \Lambda_1^0 & \Lambda_2^0 \\ a^1 & \Lambda_0^1 & \Lambda_1^1 & \Lambda_2^1 \\ \frac{a^+}{2} - a^- & \Lambda_0^2 & \Lambda_1^2 & \Lambda_2^2 \end{pmatrix}, \tag{4.4}$$

where the Λ^μ_ν are the matrix elements of the Lorentz subgroup (whose generators are E_1, F_1 and K_2) satisfying the pseudo-orthogonality condition:

$$\Lambda^\mu_\nu \Lambda^\rho_\sigma \eta^{\nu\sigma} = \eta^{\mu\rho}, \quad (\eta^{\mu\rho}) = \mathrm{diag}\,(1,-1,-1). \tag{4.5}$$

The Poisson brackets of the coordinate functions on the Poincaré group are obtained by calculating the Poisson bivector

$$\{D(g)\dot{\otimes}D(g)\} = [r, D(g)\dot{\otimes}D(g)], \tag{4.6}$$

writing the r-matrix (4.1) in terms of the matrix representation (4.3). The final result can be summarized as follows:

$$\begin{aligned}
&\{a^+,a^1\} = -2w\,a^1, \quad \{a^+,a^-\} = -2w\,a^-, \quad \{a^1,a^-\} = 0,\\
&\{\Lambda^\mu_\nu,\Lambda^\sigma_\rho\} = 0, \qquad \nu,\mu,\rho,\sigma = 0,1,2;\\
&\{\Lambda^\mu_\nu,a^+\} = -2\delta_{\mu 0}\Lambda^2_\nu - 2\delta_{\mu 2}\Lambda^0_\nu - (\mu-1)(\nu-1) + (\Lambda^\mu_0+\Lambda^\mu_2)(\Lambda^0_\nu+\Lambda^2_\nu),\\
&\{\Lambda^\mu_\nu,a^1\} = \delta_{\mu 1}(1-\nu-\Lambda^0_\nu+\Lambda^1_\nu+\Lambda^2_\nu) + \Lambda^1_\nu(\Lambda^\mu_0+\Lambda^\mu_2-1),\\
&\{\Lambda^\mu_\nu,a^-\} = \frac{1}{2}(\mu-1)^2(\nu-1) + \frac{1}{2}(\Lambda^\mu_0+\Lambda^\mu_2)(\Lambda^0_\nu-\Lambda^2_\nu).
\end{aligned} \tag{4.7}$$

It is worth comparing these expressions with the results related to the classical r-matrix of the κ-Poincaré algebra [20, 21, 22].

The classical matrix representation (4.3) is also valid for $U_w\mathcal{P}(2+1)$ since $D(P_+)^2$ vanishes. This fact can be used to get an explicit expression for $\mathcal{R}_w$:

$$D(\mathcal{R}_w) = I\otimes I + 2w(D(K_2)\wedge D(P_+) + D(E_1)\wedge D(P_1)), \tag{4.8}$$

where I is the 4×4 identity matrix. The fulfillment of property (2.6) allows to apply the FRT method [23]:

$$RT_1T_2 = T_2T_1R, \tag{4.9}$$

where R is (4.8), $T_1 = T\otimes I$, $T_2 = I\otimes T$, being T the group element (4.4) but now with non-commutative entries: $\hat{\Lambda}^\mu_\nu$ and $\hat{a}^i$. The commutation relations of the quantum Poincaré group read

$$\begin{aligned}
&[\hat{a}^+,\hat{a}^1] = -2w\,\hat{a}^1, \quad [\hat{a}^+,\hat{a}^-] = -2w\,\hat{a}^-, \quad [\hat{a}^1,\hat{a}^-] = 0,\\
&[\hat{\Lambda}^\mu_\nu,\hat{\Lambda}^\sigma_\rho] = 0, \qquad \nu,\mu,\rho,\sigma = 0,1,2;\\
&[\hat{\Lambda}^\mu_\nu,\hat{a}^+] = -2\delta_{\mu 0}\hat{\Lambda}^2_\nu - 2\delta_{\mu 2}\hat{\Lambda}^0_\nu - (\mu-1)(\nu-1) + (\hat{\Lambda}^\mu_0+\hat{\Lambda}^\mu_2)(\hat{\Lambda}^0_\nu+\hat{\Lambda}^2_\nu),\\
&[\hat{\Lambda}^\mu_\nu,\hat{a}^1] = \delta_{\mu 1}(1-\nu-\hat{\Lambda}^0_\nu+\hat{\Lambda}^1_\nu+\hat{\Lambda}^2_\nu) + \hat{\Lambda}^1_\nu(\hat{\Lambda}^\mu_0+\hat{\Lambda}^\mu_2-1),\\
&[\hat{\Lambda}^\mu_\nu,\hat{a}^-] = \frac{1}{2}(\mu-1)^2(\nu-1) + \frac{1}{2}(\hat{\Lambda}^\mu_0+\hat{\Lambda}^\mu_2)(\hat{\Lambda}^0_\nu-\hat{\Lambda}^2_\nu),
\end{aligned} \tag{4.10}$$

with the additional relations:

$$\hat{\Lambda}^\mu_\nu \hat{\Lambda}^\rho_\sigma \eta^{\nu\sigma} = \eta^{\mu\rho}, \quad (\eta^{\mu\rho}) = \mathrm{diag}\,(1,-1,-1). \tag{4.11}$$

As it happened with the κ-Poincaré group [21, 22] these commutation relations are also a Weyl quantization $[\,,\,] \to w^{-1}[\,,\,]$ of the Poisson brackets of the coordinate functions on the Poincaré group (4.7), and moreover, all the Lorentz coordinates $\hat{\Lambda}^\mu_\nu$

commute among themselves so that there is no ordering ambiguity. The associated coproduct, counit and antipode can be deduced from relations $\Delta(T) = T\dot{\otimes}T$, $\epsilon(T) = I$ and $\gamma(T) = T^{-1}$, respectively. In particular, the coproduct is:

$$\begin{aligned}
\Delta(\hat{a}^+) &= \hat{a}^+ \otimes 1 + \frac{1}{2}(\hat{\Lambda}_0^0 + \hat{\Lambda}_0^2 + \hat{\Lambda}_2^0 + \hat{\Lambda}_2^2) \otimes \hat{a}^+ \\
&\quad + (\hat{\Lambda}_1^0 + \hat{\Lambda}_1^2) \otimes \hat{a}^1 + (\hat{\Lambda}_0^0 + \hat{\Lambda}_0^2 - \hat{\Lambda}_2^0 - \hat{\Lambda}_2^2) \otimes \hat{a}^-, \\
\Delta(\hat{a}^1) &= \hat{a}^1 \otimes 1 + \frac{1}{2}(\hat{\Lambda}_0^1 + \hat{\Lambda}_2^1) \otimes \hat{a}^+ + \hat{\Lambda}_1^1 \otimes \hat{a}^1 + (\hat{\Lambda}_0^1 - \hat{\Lambda}_2^1) \otimes \hat{a}^-, \qquad (4.12) \\
\Delta(\hat{a}^-) &= \hat{a}^- \otimes 1 + \frac{1}{4}(\hat{\Lambda}_0^0 - \hat{\Lambda}_0^2 + \hat{\Lambda}_2^0 - \hat{\Lambda}_2^2) \otimes \hat{a}^+ \\
&\quad + \frac{1}{2}(\hat{\Lambda}_1^0 - \hat{\Lambda}_1^2) \otimes \hat{a}^1 + \frac{1}{2}(\hat{\Lambda}_0^0 - \hat{\Lambda}_0^2 - \hat{\Lambda}_2^0 + \hat{\Lambda}_2^2) \otimes \hat{a}^-, \\
\Delta(\hat{\Lambda}_\nu^\mu) &= \hat{\Lambda}_\sigma^\mu \otimes \hat{\Lambda}_\nu^\sigma.
\end{aligned}$$

The quantum (2+1) Poincaré plane of coordinates $(\hat{x}^+, \hat{x}^1, \hat{x}^-)$ characterized by

$$[\hat{x}^+, \hat{x}^1] = -2w\,\hat{x}^1, \quad [\hat{x}^+, \hat{x}^-] = -2w\,\hat{x}^-, \quad [\hat{x}^1, \hat{x}^-] = 0, \qquad (4.13)$$

is easily derived from the first three commutators of (4.10). Note that it includes, as a particular case, the quantum (1+1) Poincaré plane $[\hat{x}^+, \hat{x}^-] = -2w\,\hat{x}^-$ [11]. The coordinates $(\hat{x}^+, \hat{x}^1)$ could be interpreted as the parameters of a quantum light-like plane while the remaining one $\hat{x}^-$ would be a quantum time.

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